Legendre's function Polynomial of first Kind:-

The differential equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{--- (1)}$$

is called Legendre's differential equation. Its solution in the series of descending powers of x is given by

$$y = a_0\left[x^n - \frac{n(n-1)}{(2n-1)\cdot 2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)\cdot 2\cdot 4}x^{n-4} - \cdots\right] \quad (2)$$

Where a_0 is an arbitrary constant and n is a positive integer.

For, $a_0 = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{\lfloor n}$

the solution (2) is denoted by $P_n(x)$. Thus we see that,

$$P_n(x) = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{(2n-1)\cdot 2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n+3)\,2\cdot 4}x^{n-4} - \cdots\cdots\right]$$

Last term of $P_n(x)$:-

When n is even

$$(-1)^{n/2}\frac{n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1}{(2n-1)(2n-3)\cdots\cdots(n+1)\,2\cdot 4\cdot 6\cdots\cdots n}$$

When n is odd:-

$$\text{Last term} = (-1)^{\frac{n-1}{2}}\frac{n(n-1)\cdots\cdots 3\cdot 2\cdot 1}{(2n-1)(2n-3)\cdots\cdots(n+2)\,2\cdot 4\cdot 6\cdots\cdots(n-} x$$

QN.→ Explain why $(1-2xz+z^2)^{-\frac{1}{2}}$ is known as the generating function of Legendre Polynomials.

or, QN.→ Show that $(1-2xz+z^2)^{-\frac{1}{2}}$, is the generating function of Legendre Polynomials $P_n(x)$.

or, QN.→ Generating function for $P_n(x)$.

or, QN.→ Show that Legendres Polynomials $P_n(x)$ is the Coefficient of z^n in the expansion of $(1-2xz+z^2)^{-\frac{1}{2}}$ is ascending Powers of z.

Solⁿ :- We have

$(1-2xz+z^2)^{-\frac{1}{2}}$

$=\{1-z(2x-z)\}^{-\frac{1}{2}}$

$=1+\frac{1}{2}z(2x-z)+\frac{1\cdot 3}{2\cdot 4}z^2(2x-z)^2+\cdots\cdots$

$\cdots+\frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots 2n}z^n(2x-z)^n+\cdots$ ——(1)

Now, Coefficient of z^n in $\frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots 2n}z^n(2x-z)^n$

is $\frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots 2n}\cdot 2^n x^n$

$=\frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{\lfloor n}x^n$ ——(2)

Coeff. of z^n in $\frac{1\cdot 3\cdot 5\cdots\cdots(2n-3)}{2\cdot 4\cdot 6\cdots\cdots 2(n-1)}z^{n-1}(2x-z)^{n-1}$

is $\dfrac{1\cdot3\cdot5\cdots\cdots(2n-3)}{2\cdot4\cdot6\cdots\cdots2(n-1)}\left\{-(n-1)(2x)^{n-2}\right\}$

$$= -\frac{1\cdot3\cdot5\cdots\cdots(2n-3)(n-1)}{2\cdot4\cdot6\cdots\cdots2(n-1)}\,2^{n-2}x^{n-2}$$

$$= -\frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{\lfloor n-1\cdot n}\cdot\frac{n(n-1)}{2(2n-1)}\,x^{n-2}.$$

$$= -\frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{\lfloor n}\cdot\frac{n(n-1)}{(2n-1)\cdot2}\cdot x^{n-2} \quad \text{(3)}$$

~~Coefficient of~~ z^n in $\cancel{\dfrac{1\cdot3\cdot5\cdots\cdots(2n-5)}{2\cdot4\cdot6\cdots\cdots(2n-4)}}\,\cancel{\dfrac{(n-2)(n-3)}{\lfloor 2}}$

Coefficient of z^n in $\dfrac{1\cdot3\cdot5\cdots\cdots(2n-5)}{2\cdot4\cdot6\cdots\cdots(2n-4)}\,z^{n-2}(2x-z)^{n-2}$

is $\dfrac{1\cdot3\cdot5\cdots\cdots(2n-5)}{2\cdot4\cdot6\cdots\cdots(2n-4)}\cdot\dfrac{(n-2)(n-3)}{\lfloor 2}(2x)^{n-4}.$

$$= \frac{1\cdot3\cdot5\cdots\cdots(2n-3)(2n-1)}{2^{n-2}\,1\cdot2\cdot3\cdots\cdots(n-2)(n-1)n}\;\frac{n(n-1)(n-2)(n-3)}{(2n-3)(2n-1)\lfloor 2}\,2^{n-4}x^{n-4}$$

$$= \frac{1\cdot3\cdot5\cdots\cdots(2n-3)(2n-1)}{\lfloor n}\;\frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2\cdot4}\,x^{n-4}.$$

Proceeding in the similar manner and collecting the coefficient of z^n in the expansion (1) we see that coefficient of z^n is,

$$\frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{\lfloor n}\left[x^n - \frac{n\cdot(n-1)}{(2n-1)2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-3)(2n-1)\cdot2\cdot4}x^{n-4}\cdots\cdots\right]$$

$$= P_n(x).$$

Hence, the result.

Particular Cases:-

$P_1(x) =$ Coefficient of z in the exp. (1)

$= x.$

$$P_2(x) = \frac{1 \cdot 3}{2 \cdot 4} 2^2 \cdot x^2 - \frac{1}{2}$$

$P_n(1) = 1$ for all $n.$